

PERFECT POWERS GENERATED BY THE TWISTED FERMAT CUBIC

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ABSTRACT. On the twisted Fermat cubic, an elliptic divisibility sequence arises as the sequence of denominators of the multiples of a single rational point. It is shown that there are finitely many perfect powers in such a sequence whose first term is greater than 1. Moreover, if the first term is divisible by 6 and the generating point is triple another rational point then there are no perfect powers in the sequence except possibly an l th power for some l dividing the order of 2 in the first term.

1. INTRODUCTION

A divisibility sequence is a sequence

$$W_1, W_2, W_3, \dots$$

of integers satisfying $W_n | W_m$ whenever $n | m$. The arithmetic of these has been and continues to be of great interest. Ward [41] studied a large class of recursive divisibility sequences and gave equations for points and curves from which they can be generated (see also [32]). In particular, Lucas sequences can be generated from curves of genus 0. Although Ward did not make such a distinction, sequences generated by curves of genus 1 have become exclusively known as elliptic divisibility sequences [20, 21, 24, 25] and have applications in Logic [11, 17, 18] as well as Cryptography [38]. See [36, 37] for background on elliptic curves (genus-1 curves with a point). Let $d \in \mathbb{Z}$ be cube-free and consider the elliptic curve

$$C : u^3 + v^3 = d.$$

It is sometimes said that C is a twist of the Fermat cubic. The set $C(\mathbb{Q})$ forms a group under the chord and tangent method: the (projective) point $[1, -1, 0]$ is the identity and inversion is given by reflection in the line $u = v$. Suppose that $C(\mathbb{Q})$ contains a non-torsion point P . Then we can write, in lowest terms,

$$(1) \quad mP = \left(\frac{U_m}{W_m}, \frac{V_m}{W_m} \right).$$

The sequence (W_m) is a (strong) divisibility sequence (see Proposition 3.3 in [22]). Three particular questions about divisibility sequences have received much interest:

- How many terms fail to have a primitive divisor?
- How many terms are prime?
- How many terms are a perfect power?

A primitive divisor is a prime divisor which does not divide any previous term.

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1.1. Finiteness. Bilu, Hanrot and Voutier proved that all terms in a Lucas sequence beyond the 30th have a primitive divisor [3]. Silverman showed that finitely many terms in an elliptic divisibility sequence fail to have a primitive divisor [34] (see also [39]). The Fibonacci and Mersenne sequences are believed to have infinitely many prime terms [7, 8]. The latter has produced the largest primes known to date. In [9] Chudnovsky and Chudnovsky considered the likelihood that an elliptic divisibility sequence might be a source of large primes; however, (W_m) has been shown to contain only finitely many prime terms [21]. Gezer and Bizim have described the squares in some periodic divisibility sequences [23]. Using modular techniques inspired by the proof of Fermat's Last Theorem, it was finally shown in [6] that the only perfect powers in the Fibonacci sequence are 1, 8 and 144. We will show:

Theorem 1.1. *If $W_1 > 1$ then there are finitely many perfect powers in (W_m) .*

The proof of Theorem 1.1 uses the divisibility properties of (W_m) along with a modular method for cubic binary forms given in [2]. For elliptic curves in Weierstrass form similar results have been shown in [29]. In the general case, allowing for integral points, Conjecture 1.1 in [2] would give that there are finitely many perfect powers in (W_m) .

1.2. Uniformness. What is particularly special about sequences (W_m) coming from twisted Fermat cubics is that they have yielded uniform results as sharp as some of their genus-0 analogues mentioned above. It has been shown that all terms of (W_m) beyond the first have a primitive divisor [19] and, in particular, we will make use of the fact that the second term always has a primitive divisor $p_0 > 3$ (see Section 6.2 in [19]). The number of prime terms in (W_m) is also uniformly bounded [22] and, in particular, if P is triple a rational point then all terms beyond the first fail to be prime (see Theorem 1.2 in [22]). In light of Theorem 1.1, it is natural to ask if a similar results can be achieved for perfect powers. Indeed:

Theorem 1.2. *Suppose that W_1 is even and at all primes greater than 3, P has non-singular reduction (on a minimal Weierstrass equation for C). If W_m is an l th power for some prime l then*

$$l \leq \max \{ \text{ord}_2(W_1), (1 + \sqrt{p_0})^2 \},$$

where $p_0 > 3$ is a primitive divisor of W_2 . Moreover, for fixed $l > \text{ord}_2(W_1)$ the number of l th powers in (W_m) is uniformly bounded.

Although the conditions in Theorem 1.2 appear to depend heavily on the point, in the next theorem we exploit the fact that group $C(\mathbb{Q})$ modulo the points of non-singular reduction has order at most 3 for a prime greater than 3.

Theorem 1.3. *Suppose that $6 \mid W_1$ and $P \in 3E(\mathbb{Q})$ (or P has non-singular reduction at all primes greater than 3). If W_m is an l th power for some prime l then $l \mid \text{ord}_2(W_1)$. In particular, if $\text{ord}_2(W_1) = 1$ then (W_m) contains no perfect powers.*

The conditions in Theorem 1.3 are sometimes satisfied for every rational non-torsion point on C . For example, we have

Corollary 1.4. *The only solutions to the Diophantine equation*

$$U^3 + V^3 = 15W^{3l}$$

with $l > 1$ and $\gcd(U, V, W) = 1$ have $W = 0$.

2. PROPERTIES OF ELLIPTIC DIVISIBILITY SEQUENCES

In this section the required properties of (W_m) are collected.

Lemma 2.1. *Let p be a prime. For any pair $n, m \in \mathbb{N}$, if $\text{ord}_p(W_n) > 0$ then*

$$\text{ord}_p(W_{mn}) = \text{ord}_p(W_n) + \text{ord}_p(m).$$

Proof. See equation (10) in [22]. \square

Proposition 2.2. *For all $n, m \in \mathbb{N}$,*

$$\gcd(W_m, W_n) = W_{\gcd(m, n)}.$$

In particular, for all $n, m \in \mathbb{N}$, $W_n \mid W_{nm}$.

Proof. See Proposition 3.3 in [22]. \square

Theorem 2.3 ([19]). *If $m > 1$ then W_m has a primitive divisor.*

3. THE MODULAR APPROACH TO DIOPHANTINE EQUATIONS

For a more thorough exploration see [13] and Chapter 15 in [10]. As is conventional, in what follows all newforms shall have weight 2 with a trivial character at some level N and shall be thought of as a q -expansion

$$f = q + \sum_{n \geq 2} c_n q^n,$$

where the field $K_f = \mathbb{Q}(c_2, c_3, \dots)$ is a totally real number field. The coefficients c_n are algebraic integers and f is called *rational* if they all belong to \mathbb{Z} . For a given level N , the number of newforms is finite. The modular symbols algorithm [12], implemented on **MAGMA** [4] by William Stein, shall be used to compute the newforms at a given level.

Theorem 3.1 (Modularity Theorem). *Let E/\mathbb{Q} be an elliptic curve of conductor N . Then there exists a newform f of level N such that $a_p(E) = c_p$ for all primes $p \nmid N$, where c_p is p th coefficient of f and $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$.*

Proof. This is due to Taylor and Wiles [40, 42] in the semi-stable case. The proof was completed by Breuil, Conrad, Diamond and Taylor [5]. \square

The modularity of elliptic curves over \mathbb{Q} can be seen as a converse to

Theorem 3.2 (Eichler-Shimura). *Let f be a rational newform of level N . There exists an elliptic curve E/\mathbb{Q} of conductor N such that $a_p(E) = c_p$ for all primes $p \nmid N$, where c_p is the p th coefficient of f and $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$.*

Proof. See Chapter 8 of [16]. \square

Given a rational newform of level N , the elliptic curves of conductor N associated to it via the Eichler-Shimura theorem shall be computed using **MAGMA**.

Proposition 3.3. *Let E/\mathbb{Q} be an elliptic curve with conductor N and minimal discriminant Δ_{\min} . Let l be an odd prime and define*

$$N_0(E, l) := N / \prod_{\substack{\text{primes } p \mid N \\ l \mid \text{ord}_p(\Delta_{\min})}} p.$$

Suppose that the Galois representation

$$\rho_l^E : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E[l])$$

is irreducible. Then there exists a newform f of level $N_0(E, l)$. Also there exists a prime \mathcal{L} lying above l in the ring of integers \mathcal{O}_f defined by the coefficients of f such that

$$c_p \equiv \begin{cases} a_p(E) \pmod{\mathcal{L}} & \text{if } p \nmid lN, \\ \pm(1+p) \pmod{\mathcal{L}} & \text{if } p \parallel N \text{ and } p \nmid lN_0, \end{cases}$$

where c_p is the p th coefficient of f . Furthermore, if $\mathcal{O}_f = \mathbb{Z}$ then

$$c_p \equiv \begin{cases} a_p(E) \pmod{l} & \text{if } p \nmid N, \\ \pm(1+p) \pmod{l} & \text{if } p \parallel N \text{ and } p \nmid N_0. \end{cases}$$

Proof. This arose from combining modularity with level-lowering results by Ribet [30, 31]. The strengthening in the case $\mathcal{O}_f = \mathbb{Z}$ is due to Kraus and Oesterlé [27]. A detailed exploration is given, for example, in Chapter 2 of [13]. \square

Remark 3.4. Let E/\mathbb{Q} be an elliptic curve with conductor N . Note that the exponents of the primes in the factorization of N are uniformly bounded (see Section 10 in Chapter IV of [35]). In particular, only primes of bad reduction divide N and if E has multiplicative reduction at p then $p \parallel N$.

Corollary 3.5. Keeping the notation of Proposition 3.3, if p is a prime such that $p \nmid lN_0$ and $p \mid N$ then

$$l < (1 + \sqrt{p})^{2[K_f:\mathbb{Q}]}.$$

Proof. See Theorem 37 in [13]. \square

Applying Proposition 3.3 to carefully constructed Frey curves has led to the solution of many Diophantine problems. The most famous of these is Fermat's Last theorem [42] but there are now constructions for other equations and we shall make use of those described below.

3.1. A Frey curve for cubic binary forms. Let

$$F(x, y) = t_0a^3 + t_1^2y + t_2xy^2 + t_3y^3 \in \mathbb{Z}[x, y]$$

be a separable cubic binary form. In [2] a Frey curve is given for the Diophantine equation

$$(2) \quad F(a, b) = dc^l,$$

where $\gcd(a, b) = 1$, $d \in \mathbb{Z}$ is fixed and $l \geq 7$ is prime. Define a Frey curve $E_{a,b}$ by

$$(3) \quad E_{a,b} : y^2 = x^3 + a_2x^2 + a_4x + a_6,$$

where

$$\begin{aligned} a_2 &= t_1a - t_2b, \\ a_4 &= t_0t_2a^2 + (3t_0t_3 - t_1t_2)ab + t_1t_3b^2, \\ a_6 &= t_0^2t_3a^3 - t_0(t_2^2 - 2t_1t_3)a^2b + t_3(t_1^2 - 2t_0t_2)ab^2 - t_0t_3^2b^3. \end{aligned}$$

Then $E_{a,b}$ has discriminant $16\Delta_F F(a, b)^2$. Consider the Galois representation

$$\rho_l^{a,b} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E_{a,b}[l]).$$

Theorem 3.6 ([2]). Let S be the set of primes dividing $2d\Delta_F$. There exists a constant $\alpha(d, F) \geq 0$ such that if $l > \alpha(d, F)$ and $c \neq \pm 1$ then:

- the representation $\rho_l^{a,b}$ is irreducible;
- at any prime $p \notin S$ dividing $F(a,b)$ the equation (3) is minimal, the elliptic curve $E_{a,b}$ has multiplicative reduction and $l \mid \text{ord}_p(\Delta_{\min}(E_{a,b}))$.

3.2. Recipes for Diophantine equations with signature (l, l, l) . The following recipe due to Kraus [28] is taken from [10]. Consider the equation

$$Ax^l + By^l + Cz^l = 0,$$

with non-zero pairwise coprime terms and $l \geq 5$ prime. Setting $R = ABC$ assume that any prime q satisfies $\text{ord}_q(R) < l$. Without loss of generality also assume that $By^l \equiv 0 \pmod{2}$ and $Ax^l \equiv -1 \pmod{4}$. Construct the Frey curve

$$E_{x,y} : Y^2 = X(X - Ax^l)(X + By^l).$$

The conductor $N_{x,y}$ of $E_{x,y}$ is given by

$$N_{x,y} = 2^\alpha \text{rad}_2(Rxyz),$$

where

$$\alpha = \begin{cases} 1 & \text{if } \text{ord}_2(R) \geq 5 \text{ or } \text{ord}_2(R) = 0, \\ 1 & \text{if } 1 \leq \text{ord}_2(R) \leq 4 \text{ and } y \text{ is even,} \\ 0 & \text{if } \text{ord}_2(R) = 4 \text{ and } y \text{ is odd,} \\ 3 & \text{if } 2 \leq \text{ord}_2(R) \leq 3 \text{ and } y \text{ is odd,} \\ 5 & \text{if } \text{ord}_2(R) = 1 \text{ and } y \text{ is odd.} \end{cases}$$

Theorem 3.7 (Kraus [28]). *The Galois representation*

$$\rho_l^{x,y} : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E_{x,y}[l])$$

is irreducible and $N_0(E_{x,y}, l)$ in Proposition 3.3 is given by

$$N_0 = 2^\beta \text{rad}_2(R),$$

where

$$\beta = \begin{cases} 1 & \text{if } \text{ord}_2(R) \geq 5 \text{ or } \text{ord}_2(R) = 0, \\ 0 & \text{if } \text{ord}_2(R) = 4, \\ 1 & \text{if } 1 \leq \text{ord}_2(R) \leq 3 \text{ and } y \text{ is even,} \\ 3 & \text{if } 2 \leq \text{ord}_2(R) \leq 3 \text{ and } y \text{ is odd,} \\ 5 & \text{if } \text{ord}_2(R) = 1 \text{ and } y \text{ is odd.} \end{cases}$$

4. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Assume that $W_1 > 1$ and W_m is an l th power for some prime l . Firstly we will use the Frey curve for cubic binary forms constructed in Section 3.1 and prove the existence of a prime divisor p to which Corollary 3.5 can be applied, giving a bound for l . Let S be the set of primes dividing $27d$. By assumption, W_1 is divisible by a prime q . Lemma 2.1 gives that

$$l \leq \text{ord}_q(W_m) = \text{ord}_q(W_1) + \text{ord}_q(m).$$

Using Theorem 2.3 (or that there are only finitely many solutions to a Thue-Mahler equation), let l be large enough so that W_n is divisible by a prime $p \notin S$, where

$$n = q^{l - \text{ord}_q(W_1)}.$$

Note that we can choose this lower bound for l and p independently of m . Then, using Proposition 2.2, $p \mid W_m$. Now construct a Frey curve $E_{U,V}$ for the Diophantine equation

$$U_m^3 + V_m^3 = dW^l$$

as in Section 3.1 (in our case $F(x, y) = x^3 + y^3$) and consider the Galois representation

$$\rho_l : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E_{U,V}[l]).$$

Using Theorem 3.6, choose l larger than some constant so that p divides the conductor of $E_{U,V}$ exactly once and the primes dividing N_0 in Proposition 3.3 belong to S . Since there are finitely many newforms of level N_0 , Corollary 3.5 bounds l . Finally, for fixed l there are finitely many solutions by Theorem 1 in [14]. \square

5. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Assume that W_m is an l th power. We will derive an (l, l, l) equation (9) which does not depend on d and use the Frey curve given Section 3.2. Then, similarly to the proof of Theorem 1.1, the existence of a prime divisor p_0 will be shown which bounds l via Corollary 3.5. Since $2 \mid W_1$, by Lemma 2.1,

$$l \leq \text{ord}_2(W_m) = \text{ord}_2(W_1) + \text{ord}_2(m).$$

Assume that $l > \text{ord}_2(W_1)$. Then $\text{ord}_2(m) > 0$ so $m = 2m'$ for some m' .

A Weierstrass equation for C is

$$(4) \quad y^2 = x^3 - 2^4 3^3 d^2,$$

with coordinates $x = 2^2 3d/(u+v)$ and $y = 2^2 3^2 d(u-v)/(u+v)$. Write $x(mP) = A_m/B_m^2$ and $y(mP) = C_m/B_m^3$ in lowest terms.

Lemma 5.1 (see Corollary 3.2 in [22]). *Let $p = 2$ or 3 . then $p \mid W_m$ if and only if $p \nmid A_m$.*

The discriminant of (4) is $-2^{12} 3^9 d^4$ so, since d is cube free, it is minimal at any prime larger than 3 (see Remark 1.1 in Chapter VII [36]). Note that the group of points with non-singular reduction is independent of the choice of minimal Weierstrass equation. The projective equation of (4) is

$$Y^2 Z = X^3 - 2^4 3^3 d^2 Z^3.$$

Let $p > 3$ be a prime dividing d . By assumption, the partial derivatives

$$(5) \quad \frac{\partial C}{\partial X} = -3X^2, \quad \frac{\partial C}{\partial Y} = 2YZ \quad \text{and} \quad \frac{\partial C}{\partial Z} = Y^2 + 2^4 3^4 d^2 Z^2$$

do not vanish simultaneously at $P = [A_1 B_1, C_1, B_1^3]$ over the field \mathbb{F}_p . Hence, noting that $2 \nmid A_m$ from Lemma 5.1 and that non-singular points form a group, we have

$$(6) \quad \gcd(A_m^3, C_m^2) \mid 3^{3+2\text{ord}_3(d)}$$

for all m .

The inverses of the birational transformation are given by $u = (2^2 3^2 d + y)/6x$ and $v = (2^2 3^2 d - y)/6x$. Thus

$$(7) \quad \frac{U_m}{W_m} = \frac{2^2 3^2 d B_m^3 + C_m}{6 A_m B_m} \quad \text{and} \quad \frac{V_m}{W_m} = \frac{2^2 3^2 d B_m^3 - C_m}{6 A_m B_m}.$$

The assumptions made restrict the cancellation which can occur in (7) and, up to cancellation, if W_m is an l th power then so is A_m . More precisely, since W_m is an l th power and $2 \mid W_m$, Lemma 5.1 and (6) give that A_m is an l th power multiplied by a power of 3. Using the duplication formula,

$$(8) \quad \frac{A_m}{B_m^2} = \frac{A_{m'}(A_{m'}^3 + 8(2^4 3^3 d^2) B_{m'}^6)}{4 B_{m'}^2 (A_{m'}^3 - 2^4 3^3 d^2 B_{m'}^6)} = \frac{A_{m'}(A_{m'}^3 + 8(2^4 3^3 d^2) B_{m'}^6)}{4 B_{m'}^2 C_{m'}^2}.$$

Again, cancellation in (8) is restricted so $A_{m'}$ is also an l power multiplied by a power of 3. Write

$$m = 2^{\text{ord}_2(m)} n.$$

It follows that $A_n = 3^e A^l$,

$$A_n^3 + 8(2^4 3^3 d^2) B_n^6 = 3^f \bar{A}^l$$

and $C_n = \pm 3^g C^l$. Combining with $C_n^2 = A_n^3 - 2^4 3^3 d^2 B_n^6$ gives

$$(9) \quad 3^f \bar{A}^l + 2^3 3^{2g} C^{2l} = 3^{2+3e} A^{3l}.$$

Note that, by dividing (9) through by an appropriate power of 3, we can assume that 3 divides at most one of the three terms.

Let $p_0 > 3$ be a primitive divisor of W_2 . Using Proposition 2.2, $p_0 \mid W_{2n}$ and, since n is odd, $p_0 \mid \bar{A}C$. Now follow the recipe given in Section 3.2. The conductor of the Frey curve for (9) is

$$N_{\bar{A},C} = 2^3 3^\delta \text{rad}_3(\bar{A}CA)$$

and $N_0 = 2^3 3^\delta$ in Theorem 3.7, where $\delta = 0$ or 1. There is one newform

$$f = q - q^3 - 2q^5 + q^9 + 4q^{11} + \dots$$

of level $N_0 = 24$. Moreover, f is rational. Since $p_0 \mid N_{\bar{A},C}$ and $p_0 \nmid N_0$,

$$l < (1 + \sqrt{p_0})^2$$

by Corollary 3.5. Finally, for fixed $l > 1$ there are finitely many solutions to (9) (see Theorem 2 in [14]) and they are independent of d . \square

6. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. As in the proof of Theorem 1.2, consider $x(P) = A_P/B_P^2$ and $y(P) = C_P/B_P^3$ on the Weierstrass equation

$$y^2 = x^3 - 2^4 3^3 d^2$$

for C . Since P is triple another rational point, a prime of bad reduction greater 3 does not divide A_P (see Section 3 in [19]). Thus the partial derivatives (5) do not vanish simultaneously at P and so at all primes greater than 3, P has non-singular reduction on a minimal Weierstrass for C .

Now follow the proof of Theorem 1.2 up to (8). Factorizing over $\mathbb{Z}[\sqrt{-3}]$ gives

$$A_n^3 = C_n^2 + 2^4 3^3 d^2 B_n^6 = (C_n + 2^2 3 d B_n^3 \sqrt{-3})(C_n - 2^2 3 d B_n^3 \sqrt{-3}).$$

We have

$$C_n + 2^2 3 d B_n^3 \sqrt{-3} = (-1 + \sqrt{-3})^s (a + b\sqrt{-3})^3 / 2^{s+3},$$

where $s = 0, 1$ or 2 and a, b are integers of the same parity. If $s = 0$ then

$$2^3 (C_n + 2^2 3 d B_n^3 \sqrt{-3}) = a(a^2 - 9b^2) + 3b(a^2 - b^2)\sqrt{-3},$$

so

$$(10) \quad 2^3 C_n = a(a^2 - 9b^2),$$

$$(11) \quad 2^5 d B_n^3 = b(a^2 - b^2),$$

$$(12) \quad 2^2 A_n = a^2 + 3b^2.$$

If $s = 1$ then

$$\begin{aligned} 2^4 C_n &= -a^3 + 9ab^2 - 9a^2b + 9b^3, \\ 2^6 3dB_n^3 &= a^3 - 3a^2b - 9ab^2 + 3b^3, \\ 2^2 A_n &= a^2 + 3b^2. \end{aligned}$$

If $s = 2$ then

$$\begin{aligned} 2^5 C_n &= -2a^3 + 18a^2b + 18ab^2 - 18b^3, \\ 2^7 3dB_n^3 &= -2a^3 - 6a^2b + 18ab^2 + 6b^3, \\ 2^2 A_n &= a^2 + 3b^2. \end{aligned}$$

By Lemma 5.1, $6 \nmid A_n$ so we are in the case $s = 0$.

Suppose that W_m is a square. Then, from (8), $C_n = \pm C^2$, $2B_n = \pm B^2$ and $A_n = A^2$. Since $\gcd(a, b) \mid 2^2$, one of b or $a^2 - b^2$ is coprime with the odd primes dividing d . If it is b then multiplying (10) and (12) gives

$$\pm 2^5 (AC)^2 = a^5 - 6a^3b^2 - 27ab^4$$

and, since b , up to sign, is either a square or 2 multiplied by a square, dividing by b^5 gives a rational point on the hyperelliptic curve

$$Y^2 = X^5 - 6X^3 - 27X$$

with non-zero coordinates; but computations implemented in **MAGMA** confirm that the Jacobian of the curve has rank 0 and, via the method of Chabauty, there are no such points. If $a^2 - b^2$ is coprime with the odd primes dividing d then multiplying with (12) gives a rational point on the elliptic curve

$$\pm Y^2 = X^4 + 2X^2 - 3$$

or on the elliptic curve

$$\pm 2^3 Y^2 = X^4 + 2X^2 - 3$$

with non-zero coordinates; but there are no such points.

Suppose that W_m is an l th power for some odd prime l . Then, from (8), C_n , $2B_n$ and A_n are l th powers. If a is odd then (10) gives $a = C^l$, $a^2 - 9b^2 = 2^3 \bar{C}^l$ and

$$(13) \quad C^{2l} - 2^3 \bar{C}^l = 9b^2.$$

If a is even then $a = 2C^l$, $a^2 - 9b^2 = 2^2 \bar{C}^l$ and

$$(14) \quad 2^2 C^{2l} - 2^2 \bar{C}^l = 9b^2.$$

Thus, Theorem 15.3.4 in [10] (due to Bennett and Skinner [1], Ivorra [26] and Siksek [33]) and Theorem 15.3.5 in [10] (due to Darmon and Merel [15]) give that $l \leq 5$. If $l = 3$ then we have a rational point on the elliptic curve

$$Z^6 + X^3 = Y^2;$$

this curve has rank and gives a possible solution $\bar{C} = -1$, $a = C = \pm 1$ and $b = \pm 1$, but, from (11), we would have $B_n = 0$. If $l = 5$ then we have a rational point on the hyper elliptic curve

$$Y^2 = 8^e X^5 + 1,$$

where $e = 0$ or 1 ; but computations implemented in **MAGMA** confirm, via the method of Chabauty, that no such points give a required solution. \square

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